Electromagnétisme - mai 2012 - L2

I

1) $\begin{cases} \dfrac{\partial \rho(t)}{\partial t} + \mathrm{div}\,\vec{J} = 0 & , \quad \vec{J} = \gamma \vec{E} \\ \mathrm{div}\,\vec{E} = \rho/\varepsilon_0 = \mathrm{div}\left(\dfrac{\vec{J}}{\gamma}\right) \end{cases}$

$$\frac{\partial \rho(t)}{\partial t} + \gamma\,\mathrm{div}\,\vec{E} = \frac{\partial \rho(t)}{\partial t} + \frac{\gamma}{\varepsilon_0}\rho(t) = 0$$

$$\frac{d\rho(t)}{\rho(t)} = -\frac{\gamma}{\varepsilon_0}\,dt \Rightarrow \rho(t) = A\,e^{-t\varepsilon_0/\gamma}$$

$$\rho(t=0) = \rho_0 \Rightarrow \boxed{\rho(t) = \rho_0\,e^{-\varepsilon_0/\gamma\, t}}$$

ou

$$\rho(t=t_0) = \rho_0 \Rightarrow \rho(t) = \rho_0\,e^{-\varepsilon_0/\gamma\,(t-t_0)}$$

2) $\dfrac{\gamma}{\varepsilon_0} = 6{,}7\,10^{18} >> 1 \Rightarrow \tau \simeq 1{,}5\,10^{-19}\,s$!!

$\rho(t) = \rho_0\,e^{-6{,}7\,10^{18}(t-t_0)}$ → 0 instantanément

⇒ $\rho(t) = 0$ pour un conducteur métallique

↳ reste neutre en volume

3) $$r = \frac{\|\vec{J}\|}{\|\vec{J}_D\|} = \frac{\|\gamma\vec{E}\|}{\left\|\varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}\right\|} = \frac{\gamma E_0}{\varepsilon_0 E_0 \omega} = \frac{\gamma}{\varepsilon_0 \omega}$$

$\nu = 1\,MHz$ $\quad \omega = 2\pi 10^6\,s^{-1}$ $\quad r \simeq 1.10^{12} >> 1$

J_D est négligeable devant J

4) $\dfrac{\gamma}{\varepsilon_0 \omega} >> 1$ $\qquad \dfrac{\gamma}{\varepsilon_0} = \omega_{max} >> \omega$

$\omega_{max} = 6{,}7\,10^{18}\,s^{-1}$

$\nu_{max} = 1\,10^{+18}\,Hz$

$\nu << 1\,10^{18}\,Hz$ "ARQS" pour 1 MHz

5) $\mathrm{div}\,\vec{E} = 0$, $\vec{\mathrm{Rot}}\,\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$, $\mathrm{div}\,\vec{B} = 0$

$\vec{\mathrm{Rot}}\,\vec{B} = \mu_0 \vec{J}$

②

6)
$$\frac{d\mathcal{E}_{em}}{dt} = -\oiint_S \vec{R}.\vec{n}\,dS - \iiint_V \vec{J}.\vec{E}\,dV$$

$$\vec{R} = \vec{E}\wedge\frac{\vec{B}}{\mu_0} = (E_0\cos\omega t)\left(\frac{\cancel{\mu_0}\gamma E_0\cos\omega t\, R}{2\cancel{\mu_0}}\right)(\vec{e}_z\wedge\vec{e}_\varphi)$$

$$\vec{R} = -E_0^2\cos^2\omega t\,\frac{\cancel{a}\gamma}{2}\,\vec{e}_\rho = -\frac{E_0 J a}{2}\cos\omega t\,\vec{e}_\rho$$

$$-\oiint_S \vec{R}.\vec{n}\,dS = E_0^2\cos^2\omega t\,\frac{\gamma a}{2}\oiint \cancel{\rho}\,dS = E_0^2\cos^2\omega t\,\frac{\gamma}{2}\,2\pi a^2 L$$
$$\boxed{= \frac{E_0 J a}{2}\cos\omega t\ 2\pi a L}$$

$$-\iiint_V \vec{J}.\vec{E}\,dV = -\gamma E_0^2\cos^2\omega t\iiint_V dV = -\gamma E_0^2\cos^2\omega t\ \pi a^2\times L$$

$$\frac{d\mathcal{E}_{em}}{dt} = 0 \Rightarrow \oiint_S \vec{R}(-\vec{n})\,dS = \iiint_V \vec{J}.\vec{E}\,dV$$

$$\iiint_V \vec{J}.\vec{E}\,dV = \frac{J^2}{\gamma}\pi a^2\times L = \frac{I^2}{\gamma\pi^2 a^4}\pi a^2\times L$$
$$= \frac{I^2 L}{\gamma\pi a^2} = R I^2$$

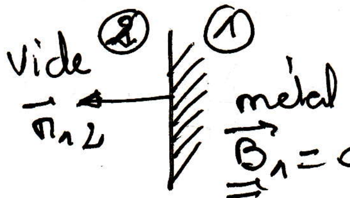
$$\boxed{R = \frac{1}{\gamma}\frac{L}{S}}$$

II

1) onde EM se propageant dans la direction z et polarisée suivant la direction x
onde transverse électrique ($E_z = 0$)

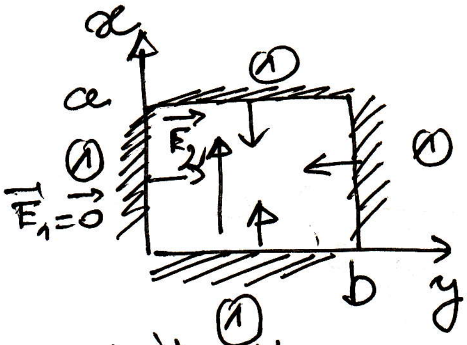
2) $\mathrm{div}\,\vec{E} = 0$ $\qquad \frac{\partial E_x}{\partial x} + \cancel{\frac{\partial E_y}{\partial y}} + \cancel{\frac{\partial E_z}{\partial z}} = 0$

$E_z = E_y = 0$ $\qquad \vec{E}$ indépendant de x $\qquad \boxed{\frac{\partial E_x}{\partial x} = 0}$

3)


$\vec{B}_1 = 0$
$\vec{E}_1 = 0$

$\vec{n}_{12}.\vec{B}_2 = 0$
$\vec{n}_{12}\wedge\vec{E}_2 = \vec{0}$



$(\leftarrow)(\vec{\Pi}_{12})$

$\vec{E} // Ox$

parois $\begin{cases} x = 0 \text{ et } x = a \\ y = 0 \text{ et } y = b \end{cases}$

$\vec{E}_2$ doit être // aux parois

continuité de la composante tangentielle de $\vec{E}$: E_T

$\begin{cases} \vec{E}_2 = \vec{0} \text{ en } \cancel{x} = 0 \text{ et } \cancel{x} = \cancel{a} \\ \vec{E}_2 = \vec{0} \text{ en } y = 0 \text{ et } y = b \end{cases}$

$\left(\sin \frac{\pi y}{b}\right)$

4) $\vec{Rot}\,\vec{Rot}\,\vec{E} = \vec{grad}\,(\operatorname{div}\vec{E}) - \Delta\vec{E} = -\Delta\vec{E}$

$\vec{Rot}\left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{Rot}\,\vec{B}) = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$

$$\boxed{\vec{\Delta E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0}}$$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = 0$$

$$\left[-\frac{\pi^2}{b^2} - k^2 + \frac{\omega^2}{c^2}\right] E_x = 0$$

équation de dispersion

$$\boxed{k^2 = \frac{\omega^2}{c^2} - \frac{\pi^2}{b^2}}$$

$k > 0$ $\qquad k^2 = \frac{\omega^2}{c^2}\left[1 - \frac{\pi^2 c^2}{b^2 \omega^2}\right]$

$\omega_c^2 = \frac{\pi^2 c^2}{b^2}$, $\quad \omega_c = \frac{\pi c}{b}$

$k^2 = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_c^2}{\omega^2}\right] \Rightarrow k = \frac{\omega}{c}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$

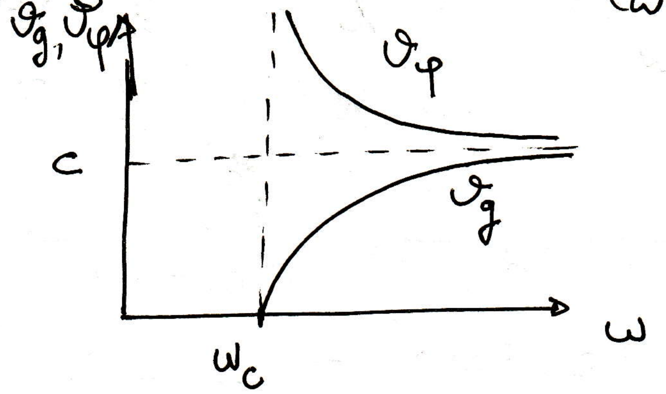
$k > 0 \Rightarrow 1 - \frac{\omega_c^2}{\omega^2} > 0 \qquad 1 > \frac{\omega_c^2}{\omega^2} \Rightarrow \omega > \omega_c$

condition de propagation

6) $v_g = \frac{d\omega}{dk}$ $\qquad v_\varphi = \frac{\omega}{k}$

$$v_\varphi = \omega \frac{c}{\omega} \frac{1}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}} = \frac{c}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}} > c$$

$$v_g = (v_\varphi)^{-1} c^2 = c\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2} < c$$



7) $\overrightarrow{\text{Rot}}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \wedge \begin{pmatrix} E_x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ +\frac{\partial E_x}{\partial z} \\ -\frac{\partial E_x}{\partial y} \end{pmatrix} = -i\omega \begin{pmatrix} 0 \\ \underline{B}_y \\ \underline{B}_z \end{pmatrix}$$

$$\begin{cases} +\frac{\partial E_x}{\partial z} = -i\omega \underline{B}_y \Rightarrow \underline{B}_y = \frac{1}{-i\omega}\frac{\partial E_x}{\partial z} = \frac{k E_0}{\omega} \sin\left(\frac{\pi y}{b}\right) e^{i(\omega t - kz)} \\ -\frac{\partial E_x}{\partial y} = -i\omega \underline{B}_z \Rightarrow \underline{B}_z = \frac{1}{i\omega}\frac{\partial E_x}{\partial y} = \frac{\pi}{ib}\frac{E_0}{\omega} \cos\left(\frac{\pi y}{b}\right) e^{i(\omega t - kz)} \end{cases}$$

$$B_y = \frac{k E_0}{\omega} \sin\left(\frac{\pi y}{b}\right) \cos(\omega t - kz)$$

$$\underline{B}_z = \frac{\pi E_0}{b\omega} \cos\left(\frac{\pi y}{b}\right) e^{i(\omega t - kz - \frac{\pi}{2})}$$

$$B_z = \frac{\pi E_0}{b\omega} \cos\left(\frac{\pi y}{b}\right) \sin(\omega t - kz)$$

8) $\operatorname{div} \vec{B} = 0$ $\qquad \cancel{\frac{\partial B_x}{\partial x}} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$

$B_x = 0$

$$\frac{\partial B_y}{\partial y} = \frac{k E_0}{\omega}\left(\frac{\pi}{b}\right) \cos\left(\frac{\pi y}{b}\right) \cos(\omega t - kz)$$

$$\frac{\partial B_z}{\partial z} = \frac{k\pi E_0}{b\omega} \cos\left(\frac{\pi y}{b}\right) \cos(\omega t - kz)$$

continuité de la composante normale de $\vec{B}$: B_N

$\vec{B}$ doit être parallèle aux parois (ou nul)

$\vec{B_2}$ // aux parois en $x = 0$ et $x = a$ B

$\vec{B}_2 = \vec{0}$ en $y = b$ et $y = 0$ $(B_y = 0)$

$(\vec{B}_2 \perp Ox)$

9) $\vec{E} \perp Oz$ $(E_y = E_z = 0)$

mais $\vec{B}$ $(B_x = 0$ et $B_z \neq 0)$

onde transverse électrique mais pas transverse magnétique

pas TEM

$\boxed{\text{Partie B}}$

1) $$\vec{R} = \frac{\vec{E} \wedge \vec{B}}{\mu_0} = \frac{1}{\mu_0}\begin{pmatrix} E_x \\ 0 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ B_y \\ B_z \end{pmatrix} = \mu_0 \begin{pmatrix} 0 \\ -E_x B_z \\ E_x B_y \end{pmatrix}$$

$$\vec{R} = \frac{1}{\mu_0}\left[(-E_x B_z)\vec{e_y} + (E_x B_y)\vec{e_z}\right]$$

$$= \frac{1}{\mu_0}\left[-\frac{E_0^2 \pi}{b\omega} \sin\left(\frac{\pi y}{b}\right)\cos\left(\frac{\pi y}{b}\right)\cos(\omega t - kz)\sin(\omega t - kz)\vec{e_y} + \frac{E_0^2 k}{\omega}\sin^2\left(\frac{\pi y}{b}\right)\cos^2(\omega t - kz)\vec{e_z}\right]$$

$$\langle \vec{R} \rangle_T = \left\langle \frac{1}{\mu_0} \frac{E_0^2 k}{\omega} \sin^2\left(\frac{\pi y}{b}\right)\cos^2(\omega t - kz)\right\rangle \vec{e_z}$$

$$= \frac{k E_0^2}{2\omega\mu_0}\sin^2\left(\frac{\pi y}{b}\right)\vec{e_z} = \frac{E_0^2 k \varepsilon_0 c^2}{2\omega}\sin^2\left(\frac{\pi y}{b}\right)\vec{e_z}$$

$$\langle P \rangle_T = \iint_S \vec{R}\cdot\vec{n}\, dS = \int_0^a \int_0^b R\,\vec{e_z}\,\vec{e_z}\, dx\, dy$$

$$\langle P \rangle_T = \frac{k E_0^2 a}{2\omega\mu_0}\int_0^b \sin^2\left(\frac{\pi y}{b}\right) dy = \frac{k E_0^2 ab}{4\omega\mu_0} = \frac{k\varepsilon_0 c^2 ab E_0^2}{4\omega}$$

~~3) $\left\langle \frac{dE_{em}}{dz} \right\rangle_T = \frac{\varepsilon_0 E_0^2 ab}{4} = \ldots$~~

~~$(J.m^{-1}) = \langle P \rangle_T \frac{k}{\omega} \ldots = \langle P \rangle_T \ldots \; W.m.s^{-1}$~~

3) $$\frac{\langle \mathcal{E}_{em} \rangle_T}{dz} = \frac{\varepsilon_0 E_0^2 ab}{4} = \frac{\langle P \rangle_T \, \varepsilon_0 \mu_0 \omega}{k}$$

$$= \langle P \rangle_T \frac{\omega}{k c^2} = \langle P \rangle_T \frac{\omega}{k v_g v_\varphi}$$

$$\frac{\langle \mathcal{E}_{em} \rangle_T}{dz} = \frac{\langle P \rangle_T}{v_g}$$

$$\langle P \rangle_T = v_g \frac{\langle \mathcal{E}_{em} \rangle_T}{dz}$$

$$(W) \equiv (m\,s^{-1}) \times (J\,m^{-1})$$

$$v_g = \frac{\langle P \rangle_T}{\langle \mathcal{E}_{em} \rangle_T / dz}$$